A fractal local smoothing problem for the wave equation

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ABSTRACT. For any given set $E \subset [1,2]$, we discuss a fractal frequency-localized version of the L^p local smoothing estimates for the half-wave propagator with times in E. A conjecture is formulated in terms of a quantity involving the Assouad spectrum of E and the Legendre transform. We validate the conjecture for radial functions. We also prove a similar result for fractal-time $L^2 \to L^q$ and square function bounds, for arbitrary L^2 functions and general time sets. We formulate a conjecture for $L^p \to L^q$ generalizations.

1. INTRODUCTION

Consider the half-wave propagator

$$e^{it\sqrt{-\Delta}} f(x) = \frac{1}{(2\pi)^d} \int_{\mathbb{R}^d} \widehat{f}(\xi) e^{i\langle x,\xi\rangle + it|\xi|}\, \mathrm{d}\xi, \qquad x \in \mathbb{R}^d,\ t > 0,$$

initially defined for Schwartz functions f, where $\widehat{f}(\xi) = \int_{\mathbb{R}^d} f(y) e^{-i\langle y,\xi\rangle} \mathrm{d}y$ denotes the Fourier transform. It is well-known since the work of Miyachi [Miy80] and Peral [Per79] that for fixed time $t > 0$ and $2 \le p < \infty$, there exists a locally bounded constant $C_t > 0$ such that

$$\|e^{it\sqrt{-\Delta}} f\|_{L^p(\mathbb{R}^d)} \le C_t \|f\|_{L^p_{s_p}(\mathbb{R}^d)}, \qquad s_p = (d-1)\big(\tfrac{1}{2} - \tfrac{1}{p}\big). \tag{1.1}$$

Here L^p_s denotes the usual L^p-Sobolev space. The result is sharp in the sense that s_p cannot be replaced by a smaller number. The local smoothing problem for the wave equation, proposed by Sogge [Sog91], aims to establish sharp space-time L^p-Sobolev estimates for $e^{it\sqrt{-\Delta}}$ where $t \in [1,2]$. In particular, one aims to gain derivatives over (1.1) and conjectures that for all $2 < p < \infty$ and all $\varepsilon > 0$ there exists a constant $C_\varepsilon > 0$ such that

$$\Big(\int_1^2 \|e^{it\sqrt{-\Delta}} f\|_p^p \, \mathrm{d}t \Big)^{\frac{1}{p}} \le C_\varepsilon \|f\|_{L^p_{\sigma_p+\varepsilon}}, \quad \sigma_p = \begin{cases} 0 & \text{if } 2 < p < \frac{2d}{d-1} \\ s_p - \frac{1}{p} & \text{if } p > \frac{2d}{d-1} \end{cases}. \tag{1.2}$$

The first result of this kind was proved by Wolff [Wol00] for large values of p. In two dimensions, Sogge's conjecture was recently established for all $2 < p < \infty$ by Guth, Wang and Zhang [GWZ20]. In [HNS11] it was also conjectured that for $p > \frac{2d}{d-1}$ the inequality (1.2) should hold even with $\varepsilon = 0$, and this endpoint result was verified for $p > \frac{2(d-1)}{d-3}$, $d \ge 4$. For $d \ge 3$, the current best result with the ε-loss

corresponds to the range $p \geq \frac{2(d+1)}{d-1}$, which was proved by Bourgain and Demeter [BD15]. We refer to the survey [BHS21] for further history of the problem. The L^p-bound also implies an inequality with $L^p_{\mathrm{rad}}(L^2_{\mathrm{sph}})$ in place of $L^p(\mathbb{R}^d)$; the version of Sogge's conjecture in this category was proved in [MS01].

In this paper, we introduce a *fractal* version of the local smoothing problem, which we validate in the radial case. To formulate it, we first note that (1.2) can be rewritten in a discretized form when frequency localized to an annulus of frequencies $\approx 2^j$ where $j \geq 1$. Define $P_j = \varphi(2^{-j}|D|)$ where φ is a smooth bump function supported in the interval $(\frac{1}{4}, 4)$. Then a version equivalent to (1.2) with $\varepsilon > 0$ is that for all $2 < p < \infty$ and $s > \max(s_p, \frac{1}{p})$, there exists a constant $C_s > 0$ such that

$$\Big(\sum_{t \in E_j} \|e^{it\sqrt{-\Delta}} P_j f\|_p^p \Big)^{\frac{1}{p}} \leq C_s 2^{js} \|f\|_p \tag{1.3}$$

where E_j is a maximal 2^{-j}-separated subset of $[1,2]$. In the fractal problem we replace $[1,2]$ by an *arbitrary* subset E, and let E_j be a 2^{-j}*-discretization of* E, i.e., a maximal 2^{-j}-separated subset of E. We then ask how the optimal exponent s is determined by E.

Given any bounded $E \subset \mathbb{R}$ define the *Legendre–Assouad function* $\nu_E^\sharp \colon \mathbb{R} \to \mathbb{R}$ by

$$\nu_E^\sharp(\alpha) = \limsup_{\delta \to 0} \frac{\log\big(\sup_{\delta \leq |I| \leq 1} |I|^{-\alpha} N(E \cap I, \delta)\big)}{\log(\frac{1}{\delta})}, \tag{1.4}$$

where the supremum is taken over all intervals I of length between δ and 1. The terminology in this definition is motivated by Theorem 1.2 below. The quantity $\nu_E^\sharp(\alpha)$ was introduced in a study of circular maximal operators by the first, second and fourth authors in [BRS24+]. It turns out that for *all* $E \subset [1,2]$, the critical exponent in (1.3) can be expressed in terms of $\nu_E^\sharp$, at least in the radial setting. While $\nu_E^\sharp(\alpha)$ is well-defined for all $\alpha \in \mathbb{R}$, we care about the case $\alpha \geq 0$, because $\nu_E^\sharp(\alpha) = \dim_{\mathrm{M}} E$ for all $\alpha \leq 0$, where $\dim_{\mathrm{M}} E$ denotes the upper Minkowski dimension of E.

Theorem 1.1. *Let $E \subset [1,2]$ and $2 \leq p < \infty$. Then for every $\varepsilon > 0$ there exist a constant $C_{\varepsilon,p} > 0$ such that for all $j \geq 1$ and all 2^{-j}-discretizations E_j of E,*

$$\Big(\sum_{t \in E_j} \|e^{it\sqrt{-\Delta}} P_j f\|_p^p \Big)^{1/p} \leq C_{\varepsilon,p} 2^{j(\frac{1}{p}\nu_E^\sharp(ps_p)+\varepsilon)} \|f\|_{L^p_{\mathrm{rad}}} \tag{1.5}$$

for all radial L^p functions f. Moreover, the inequality is sharp up to the ε-loss.

The proof of the upper bound is fairly standard: it is a refinement of the argument in [MS95] for $E = [1,2]$ (see also [CCS02]) and extends it as an essentially sharp result for arbitrary sets $E \subset [1,2]$. Note that for $E = [1,2]$ we have

$$\tfrac{1}{p}\nu_{[1,2]}^\sharp(ps_p) = \begin{cases} \frac{1}{p} & \text{if } p \leq \frac{2d}{d-1} \\ s_p & \text{if } p > \frac{2d}{d-1} \end{cases} \tag{1.6}$$

matching the exponents in the standard local smoothing conjecture (1.3). It is reasonable to conjecture that Theorem 1.1 holds for all L^p functions; this constitutes a fractal analogue of Sogge's conjecture for general L^p functions.

The Legendre–Assouad function is closely related to the *Assouad spectrum* of E, which we now recall. For $0 \le \theta < 1$ define $\dim_{\mathrm{A},\theta} E$ as the infimum over all exponents $a > 0$ for which there exists a constant C such that

$$N(E \cap I, \delta) \le C(|I|/\delta)^a.$$

for all intervals I with $|I| = \delta^\theta$ and $\delta \in (0,1)$ (Fraser–Yu [FY18]). The *Assouad spectrum* is the function $\theta \mapsto \dim_{\mathrm{A},\theta} E$. At $\theta = 0$ we recover the upper Minkowski dimension

$$\beta = \dim_{\mathrm{M}} E = \dim_{\mathrm{A},0} E.$$

The Assouad spectrum is continuous on $[0,1)$ and the limit

$$\gamma = \dim_{\mathrm{qA}} E = \lim_{\theta \to 1-} \dim_{\mathrm{A},\theta} E$$

exists and is called the *quasi-Assouad dimension* (Lü–Xi [LX16]). We refer the reader to Fraser's monograph [Fra20] for further information.

The *Legendre transform* of a (not necessarily convex) continuous function ν defined on a closed interval $I \subset \mathbb{R}$ is defined by

$$\nu^*(\alpha) = \sup_{\theta \in I} \theta\alpha - \nu(\theta), \tag{1.7}$$

which is finite for all $\alpha \in \mathbb{R}$ if I is compact. It was observed in [BRS24+] that $\nu_E^\sharp$ equals the Legendre transform of

$$\nu_E(\theta) = -(1-\theta)\dim_{\mathrm{A},\theta} E, \quad \theta \in [0,1], \tag{1.8}$$

under a certain regularity assumption on E. We note that a certain conjugate to the Assouad spectrum also appeared in [BFKR24+].

The function ν_E is increasing, but may not be convex (see [FY18; Rut24]). The next theorem says that the regularity assumption in [BRS24+] can be removed, and combining this with the characterization of Assouad spectra by the third author [Rut24] allows us to obtain a simple characterization of the class of functions which occur as Legendre–Assouad function of some subset of $[1,2]$. [1]

Theorem 1.2. *The following hold:*

(i) For all bounded $E \subset \mathbb{R}$, $\nu_E^\sharp = \nu_E^*$.

(ii) A function $\tau : [0,\infty) \to [0,\infty)$ satisfies $\nu_E^\sharp|_{[0,\infty)} = \tau$ for some bounded set $E \subset \mathbb{R}$ if and only if τ is increasing, convex, and satisfies $\tau(\alpha) = \alpha$ for $\alpha \ge 1$.

As a consequence of (i), $\nu_E^\sharp$ only depends on the convex hull of ν_E which by convex duality is equal to $\nu_E^{**} = (\nu_E^\sharp)^*$. This together with the characterization of increasing Assouad spectra in [Rut24, Corollary B] gives (ii). We provide the details in §2.

[1] The value of $\nu_E^\sharp(\alpha)$ does not change under dilations and translations, so if $\tau = \nu_E^\sharp$ for some bounded E, then also $\tau = \nu_{E'}^\sharp$ for some $E' \subset [1,2]$.

Corollary 1.3. *Let E be bounded and $\gamma = \dim_{\mathrm{qA}} E$. Then*

$$\nu_E^\sharp(\alpha) = \alpha \quad \text{if} \quad \alpha \geq \gamma \tag{1.9}$$

and the number γ is minimal with this property.

Note that (1.9) was already observed in [BRS24+]. For $0 \leq \alpha \leq \gamma$, $\nu_E^\sharp(\alpha)$ can be interpreted as a new dimensional spectrum interpolating between Minkowski and quasi-Assouad dimension. If $\beta < \gamma$, then $\nu_E^\sharp(\alpha)$ is strictly increasing for $\alpha \geq 0$.

Applying (1.9) to the sharp exponent in (1.5) we obtain $\frac{1}{p}\nu_E^\sharp(ps_p) = s_p$ for $p \geq p_\gamma = \frac{2(d-1+\gamma)}{d-1}$, where $\gamma = \dim_{\mathrm{qA}} E$. This implies that if the standard local smoothing conjecture (1.3) is known for some $p_\circ \geq \frac{2d}{d-1}$, then the corresponding fractal conjecture is also true for all $p \geq p_\circ$ and all $E \subset [1,2]$. However for $p < \frac{2d}{d-1}$, the fractal problem differs from the classical one, and for general $E \subset [1,2]$ the supercritical regime $p \geq \frac{2d}{d-1}$ in (1.6) is replaced by $p \geq \frac{2(d-1+\gamma)}{d-1}$.

In particular, we have the following.

Corollary 1.4. *For every $E \subset [1,2]$ with $\gamma = \dim_{\mathrm{qA}} E$, $p \geq p_\gamma = \frac{2(d-1+\gamma)}{d-1}$, $\varepsilon > 0$, and radial f,*

$$\Big(\sum_{t \in E_j} \|e^{it\sqrt{-\Delta}} P_j f\|_p^p \Big)^{1/p} \leq C_{\varepsilon,p}\, 2^{j(s_p+\varepsilon)} \|f\|_{L^p_{\mathrm{rad}}}.$$

The exponent is sharp up to the ε-loss.

Theorem 1.2 illustrates a striking contrast to the classical local smoothing problem: solving the fractal smoothing problem for $p = p_\gamma$ does typically *not* imply sharp estimates in the range $2 < p < p_\gamma$ by interpolating with $p = 2$.

Indeed, for this interpolation to be sharp it is necessary that

$$\tfrac{1}{p}\nu_E^\sharp(ps_p) = \begin{cases} (1-\frac{\beta}{\gamma})s_p + \frac{\beta}{p} & \text{if } 2 \leq p \leq p_\gamma, \\ s_p & \text{if } p \geq p_\gamma. \end{cases} \tag{1.10}$$

That is, $\nu_E^\sharp$ consists of two affine linear pieces. But Theorem 1.2 (ii) says in particular that the function $\nu_E^\sharp$ need not be piecewise affine. This is the same phenomenon observed in [RS23] for the $L^p \to L^q$ type sets of spherical maximal functions (although $\nu_E^\sharp$ was not mentioned explicitly there).

The interpolated exponents (1.10) occur if E is *quasi-Assouad regular*, that is if its (upper) Assouad spectrum takes the form

$$\dim_{\mathrm{A},\theta} E = \begin{cases} \frac{\beta}{1-\theta} & \text{if } 0 < \theta \leq 1 - \frac{\beta}{\gamma}, \\ \gamma & \text{if } 1 - \frac{\beta}{\gamma} \leq \theta < 1. \end{cases} \tag{1.11}$$

The equation can be interpreted as saying that the Assouad spectrum should always achieve its largest possible value given the endpoint values β and γ (indeed, $\dim_{\mathrm{A},\theta} E$ is always bounded by the right hand-side in (1.11), see [FY18]). Examples include all cases where $\beta = \gamma$ (such as self-similar Cantor-type sets), and convex

sequences $E = \{1 + n^{-a} : n \geq 1\}$ with $a > 0$, where $\beta = (a+1)^{-1}$ and $\gamma = 1$. For other examples see [RS23; Rut24]. The simplest examples where (1.10) fails are of the form $E = E_1 \cup \cdots \cup E_k$ with $E_1, \ldots, E_k$ quasi-Assouad regular.

Then

$$\nu_E^\sharp = \max(\nu_{E_1}^\sharp, \ldots, \nu_{E_k}^\sharp).$$

Remark 1.5. The above definition of quasi-Assouad regular sets is equivalent to that introduced in [RS23] (see [Rut24, Corollary C] for the equivalence).

The Legendre–Assouad function is also relevant for other related estimates with a fractal feature (e.g. circular maximal functions [BRS24+]). As a further example we prove certain Strichartz type estimates for $e^{it\sqrt{-\Delta}}$ with fractal sets of times $E \subset [1,2]$. In this case we obtain a result valid for all L^2-functions (not necessarily radial).

Theorem 1.6. *Let $E \subset [1,2]$, $2 \leq r \leq q < \infty$ and*

$$s > s_E(q) = \tfrac{d+1}{2}(\tfrac{1}{2} - \tfrac{1}{q}) + \tfrac{1}{q}\nu_E^\sharp\big(\tfrac{d-1}{2}(\tfrac{q}{2} - 1)\big).$$

Then there exists a constant $C_{s,q} > 0$ such that for all $f \in L^2$

$$\Big\| \Big(\sum_{t \in E_j} |e^{-it\sqrt{-\Delta}} P_j f|^r \Big)^{1/r} \Big\|_q \leq C_{s,q} 2^{js} \|f\|_2. \tag{1.12}$$

Moreover, if $s < s_E(q)$ this conclusion fails to hold.

The critical exponent $s_E(q)$ does not depend on r. Thus the upper bound follows from the case $r = 2$. The corresponding square function is relevant to variation bounds for spherical averages: for recent results and further references see [BOR+22], [Whe24+]. The proof of the upper bounds in Theorem 1.6 is based on familiar TT^* arguments; it can be seen as a refinement of a result in [AHRS21] for $r = q$. We note that it is mainly interesting for us in the range $2 \leq q < q_\gamma = \frac{2(d-1+2\gamma)}{d-1}$, with $\gamma = \dim_{\mathrm{qA}} E$. Since $\nu_E^\sharp(\alpha) = \alpha$ for $\alpha \geq \gamma$ and since $\frac{d-1}{2}(\frac{q}{2} - 1) \geq \gamma$ if and only if $q \geq q_\gamma$, we see that the operator norm in (1.12) is $\lesssim_{q,\varepsilon} 2^{jd(\frac{1}{2}-\frac{1}{q})+j\varepsilon}$ for $q \geq q_\gamma$. If one replaces the quasi-Assouad dimension γ by the Assouad dimension a stronger result for $q \geq q_\gamma$ can be proven with $\varepsilon = 0$: see Proposition 6.3 below.

The case $r = q$ in Theorem 1.6 is a specific case of a more general $L^p \to L^q$ fractal local smoothing conjecture.

Conjecture 1.7. *Let $E \subset [1,2]$ and $1 < p \leq q < \infty$, $q > p'$. Then for every $s > s_E(p,q) := \frac{d+1}{2}(\frac{1}{p} - \frac{1}{q}) + \frac{1}{q}\nu_E^\sharp\big(\frac{q(d-1)}{2}(1 - \frac{1}{p} - \frac{1}{q})\big)$, there exists a constant $C_{s,p,q} > 0$ such that*

$$\Big(\sum_{t \in E_j} \|e^{it\sqrt{-\Delta}} P_j f\|_q^q \Big)^{\frac{1}{q}} \leq C_{s,p,q} 2^{js} \|f\|_p$$

holds for all $j \geq 1$ and all 2^{-j}-discretizations E_j of E.

It will be shown in Proposition 4.1 that $s_E(p,q)$ cannot be replaced by a smaller value. We note that $s_E(2,q) = s_E(q)$ in Theorem 1.6, and thus the conjecture is verified for $p = 2$. For $q = p$, we note that $s_E(p,p) = \frac{1}{p}\nu_E^\sharp(ps_p)$, which matches the exponent in Theorem 1.1 and the corresponding conjecture for general L^p-functions. For $E = [1,2]$, one recovers the numerology in [BS22, Conjecture 1.1] and for $E = \{t_0\} \subset [1,2]$, it coincides with the numerology of the fixed-time $L^p - L^q$ estimate, which follows from interpolating (1.1) with the standard bound $\|e^{it\sqrt{-\Delta}}P_j\|_{L^1\to L^\infty} \lesssim 2^{j\frac{d+1}{2}}$. Note that $s_E(p,q) = s_q + \frac{1}{p} - \frac{1}{q}$ for $q \geq p'\frac{d-1+2\gamma}{d-1}$ and all $E \subseteq [1,2]$. Under this condition, the case $q = p$ of the above conjecture implies the $q > p$ case by interpolation with the $L^1 \to L^\infty$ bound, similarly to the case $E = [1,2]$. However, this is not generally true if $q < p'\frac{d-1+2\gamma}{d-1}$ and $E \subset [1,2]$ is arbitrary.

Remark 1.8. Different types of fractal space-time problems for the wave equation or spherical means have been considered in the literature, see for example [CHL17; HKL22; IKS+19; Obe06; Whe24+]. These authors put a fractal measure μ on space-time $\mathbb{R}^d \times \mathbb{R}$ and ask for corresponding $L^p(\mathbb{R}^d) \to L^q(\mu)$ estimates, using some notion of Hausdorff dimension for the measure μ. See also Wheeler [Whe24+] where Corollary 1.4 is conjectured for all $f \in L^p$ in the case when the Minkowski and Assouad dimensions of E coincide. Many interesting questions arise.

Notational conventions. Given a list of objects L and real numbers $A, B \geq 0$, here and throughout we write $A \lesssim_L B$ or $B \gtrsim_L A$ to indicate $A \leq C_L B$ for some constant C_L which depends on only items in the list L. We write $A \sim_L B$ to indicate $A \lesssim_L B$ and $B \lesssim_L A$. We say that a real-valued function f on the real line is increasing (as opposed to non-decreasing) if $f(x) \leq f(y)$ whenever $x \leq y$ are in its domain.

Structure of the paper. In §2 we discuss properties of $\nu_E^\sharp$ and prove Theorem 1.2. In §3 we prove sharpness of Theorem 1.1. In §4 we prove sharpness of Theorem 1.6 and motivate the numerology in the $L^p \to \ell^q_{E_j}(L^q)$ conjecture. In §5 we prove the upper bounds in Theorem 1.1 and in §6 we prove the upper bounds in Theorem 1.6.

2. THE LEGENDRE–ASSOUAD FUNCTION

In this section we prove Theorem 1.2 and Corollary 1.3. Let us first recall some basic facts about the Legendre transform. Let $\nu\colon I \to \mathbb{R}$ be a continuous function that is not necessarily convex, defined on a closed interval $I \subset \mathbb{R}$. Its Legendre transform ν^* (defined by (1.7)) is always convex, as a supremum of affine functions, and a basic fact is convex duality: the function $\nu^{**} = (\nu^*)^*$ is the convex hull of ν, i.e. it is the largest convex function bounded above by ν. In particular, we have $\nu = \nu^{**}$ if and only if ν is convex. Note that here we adopt the convention to extend ν to a function on $\mathbb{R}$ by setting $\nu(\theta) = \infty$ for $\theta \notin I$, so that both ν and ν^* are defined on all of $\mathbb{R}$. More details can be found in e.g. [Roc97].

Proof (of Theorem 1.2, (i)). Let $\gamma_E(\theta) = \dim_{\mathrm{A},\theta} E$ and

$$\varphi(\delta,\theta) = \frac{\sup_{|I|=\delta^\theta} \log N(E\cap I,\delta)}{(1-\theta)\log(\frac{1}{\delta})}.$$

Then the claim can be rewritten as

$$\limsup_{\delta\to 0} \sup_{\theta\in[0,1]} \theta\alpha + (1-\theta)\varphi(\delta,\theta) = \max_{\theta\in[0,1]} \theta\alpha + (1-\theta)\gamma_E(\theta). \tag{2.1}$$

Fix $\alpha \geq 0$. We first prove the lower bound in (2.1). Since the Assouad spectrum is continuous, there exists $\theta_\alpha \in [0,1]$ such that the right-hand side of (2.1) equals

$$\theta_\alpha\alpha + (1-\theta_\alpha)\gamma_E(\theta_\alpha). \tag{2.2}$$

By taking $\theta = \theta_\alpha$ in the supremum,

$$\limsup_{\delta\to 0} \sup_{\theta\in[0,1]} \theta\alpha + (1-\theta)\varphi(\delta,\theta) \geq \theta_\alpha\alpha + (1-\theta_\alpha)\limsup_{\delta\to 0}\varphi(\delta,\theta_\alpha),$$

which is equal to (2.2), concluding the proof of the lower bound.

To prove the upper bound let $\varepsilon > 0$. Let (δ_n) be a monotone sequence converging to zero so that the left-hand side of (2.1) equals

$$\lim_{n\to\infty} \sup_{\theta\in[0,1]} \theta\alpha + (1-\theta)\varphi(\delta_n,\theta).$$

By definition of the supremum, for every $n \in \mathbb{N}$ there exists $\theta_n \in [0,1]$ such that

$$\sup_{\theta\in[0,1]} \theta\alpha + (1-\theta)\varphi(\delta_n,\theta) \leq \theta_n\alpha + (1-\theta_n)\varphi(\delta_n,\theta_n) + \varepsilon. \tag{2.3}$$

By passing to a subsequence we may assume that (θ_n) converges to a limit $\theta_* \in [0,1]$. By continuity of $\theta \mapsto (1-\theta)\gamma_E(\theta)$ at θ_* we may choose a value $\theta_*^- = \theta_*^-(\varepsilon)$ in $[0,\theta_*)$ close to θ_* so that

$$(1-\theta_*^-)\gamma_E(\theta_*^-) \leq (1-\theta_*)\gamma_E(\theta_*) + \varepsilon. \tag{2.4}$$

Further since $\theta_n \to \theta_*$ there exists N_ε so that for all $n \geq N_\varepsilon$ we have

$$\theta_n \geq \theta_*^- \quad \text{and} \quad \theta_n\alpha \leq \theta_*\alpha + \varepsilon. \tag{2.5}$$

Then there exists a constant $C_\varepsilon > 0$ such that

$$\sup_{|I|=\delta_n^{\theta_n}} N(E\cap I,\delta_n) \leq \sup_{|I|=\delta_n^{\theta_*^-}} N(E\cap I,\delta_n) \leq C_\varepsilon \delta_n^{-(1-\theta_*^-)\gamma_E(\theta_*^-)-\varepsilon} \tag{2.6}$$

for $n \geq N_\varepsilon$ (using $\theta_n \geq \theta_*^-$ in the first inequality and the definition of Assouad spectrum in the second). Hence, for $n \geq N_\varepsilon$,

$$(1-\theta_n)\varphi(\delta_n,\theta_n) \leq \tfrac{\log C_\varepsilon}{\log(\frac{1}{\delta_n})} + (1-\theta_*^-)\gamma_E(\theta_*^-) + \varepsilon.$$

By making N_ε larger if needed we may also assume $\log(C_\varepsilon)/\log(1/\delta_n) \le \varepsilon$ for all $n \ge N_\varepsilon$. Then from (2.4) and (2.6)

$$(1-\theta_n)\varphi(\delta_n,\theta_n) \le (1-\theta_*^-)\gamma_E(\theta_*^-) + 2\varepsilon \le (1-\theta_*)\gamma_E(\theta_*) + 3\varepsilon$$

for $n \ge N_\varepsilon$. Combining this with (2.3) and (2.5) we conclude

$$\begin{aligned}\sup_{\theta\in[0,1]} \theta\alpha + (1-\theta)\varphi(\delta_n,\theta) &\le \theta_*\alpha + (1-\theta_*)\gamma_E(\theta_*) + 5\varepsilon \\ &\le \max_{\theta\in[0,1]} \theta\alpha + (1-\theta)\gamma_E(\theta) + 5\varepsilon.\end{aligned}$$

Since $\varepsilon > 0$ was arbitrary this concludes the proof. □

Proof (of Theorem 1.2, (ii)). By part (i), $\nu_E^\sharp$ is convex, because it equals the Legendre transform of ν_E. It is increasing since $\theta \ge 0$ in the maximum in (2.1). Finally, if $\alpha \ge 1$, then $\nu_E^\sharp(\alpha) = \alpha$: the lower bound always holds by taking $\theta = 1$ and the upper bound follows from $\dim_{\mathrm{A},\theta} E \le 1$ and $\alpha \ge 1$.

To show the converse, let $\tau\colon [0,\infty) \to [0,\infty)$ be increasing, convex and $\tau(\alpha) = \alpha$ for $\alpha \ge 1$. Note that $\tau(\alpha) \ge \alpha$ for all $\alpha \in [0,\infty)$ by convexity and assumption. Define the function $\nu\colon [0,1] \to \mathbb{R}$ by

$$\nu(\theta) = \tau^*(\theta) = \sup_{\alpha\ge 0} \alpha\theta - \tau(\alpha)$$

for $0 \le \theta \le 1$. (Note $\tau^*(\theta)$ is defined for all $\theta \le 1$.) Note that $\nu(1) = 0$ since $\alpha \le \tau(\alpha)$ and $\tau(1) = 1$. For $\theta \in (0,1)$ define

$$\gamma(\theta) = \frac{-\nu(\theta)}{1-\theta}. \tag{2.7}$$

We now use the characterization of the class of increasing functions that are attainable as the Assouad spectrum of a bounded set $E \subset \mathbb{R}$ from [Rut24, Corollary B]. It states that if $\gamma\colon (0,1) \to [0,1]$ is an increasing function such that

$$\theta \mapsto \nu(\theta) = -(1-\theta)\gamma(\theta)$$

is increasing on $(0,1)$, then γ is the Assouad spectrum of a bounded subset $E \subset \mathbb{R}$.

Let us verify these assumptions for γ as defined in (2.7). First, γ is increasing because $\nu = \tau^*$ is convex; indeed, since $\nu(1) = 0$, we have $\nu(\theta t + (1-t)) \le t\nu(\theta)$ for all $t, \theta \in [0,1]$. Second, ν is increasing because the supremum in its definition is taken over $\alpha \ge 0$. Finally, γ takes values in the interval $[0,1]$: the inequality $\gamma(\theta) \ge 0$ follows because $\nu(1) = 0$ and ν is increasing, and the inequality $\gamma(\theta) \le 1$ is equivalent to $\tau^*(\theta) \ge \theta - 1$ which holds because $\tau(1) = 1$ (take $\alpha = 1$ in the supremum defining τ^*). Thus, we obtain a bounded E with $\nu_E = \nu$, where $\nu_E(\theta) = -(1-\theta)\dim_{\mathrm{A},\theta} E$ as in (1.8). By part (i) of Theorem 1.2 and convex duality,

$$\nu_E^\sharp = \nu_E^* = \nu^* = \tau^{**} = \tau$$

which concludes the proof. □

Proof (of Corollary 1.3). Let $\gamma_E(\theta) = \dim_{\mathrm{A},\theta} E$, $\gamma = \dim_{\mathrm{qA}} E$. By Theorem 1.2 (i),

$$\nu_E^\sharp(\alpha) = \sup_{\theta\in[0,1]} \alpha\theta + (1-\theta)\gamma_E(\theta). \tag{2.8}$$

First note that $\nu_E^\sharp(\alpha) \geq \alpha$ for all $\alpha \in \mathbb{R}$ by taking $\theta = 1$ in the supremum. If $\alpha \geq \gamma$, then $\nu_E^\sharp(\alpha) \leq \alpha$ holds also by using $\gamma_E(\theta) \leq \gamma$. To show the minimality claim suppose $\nu_E^\sharp(\alpha) > \alpha$. It then suffices to show that $\alpha < \gamma$. By continuity of the Assouad spectrum, for every $\alpha \in \mathbb{R}$ there exists $\theta_\alpha \in [0,1]$ where the supremum in (2.8) is attained. Since $\gamma_E(\theta_\alpha) \leq \gamma$ we obtain $\alpha < \nu_E^\sharp(\alpha) \leq \alpha\theta_\alpha + (1-\theta_\alpha)\gamma$, equivalently $(1-\theta_\alpha)\alpha < (1-\theta_\alpha)\gamma$ and $\theta_\alpha \neq 1$. Thus $\alpha < \gamma$, as required. □

3. LOWER BOUNDS IN THEOREM 1.1

In this section we test the half-wave operator on suitable radial functions to show the (essential) sharpness of Theorem 1.1. Let $I \subset [1,2]$ be an interval of length $|I| \geq M2^{-j}$ containing points in E_j, with $M \geq 1$ a sufficiently large constant chosen below. It suffices to prove that

$$\sup_{\|g\|_p\leq 1} \sum_{t\in E_j\cap I} \|e^{it\sqrt{-\Delta}}g\|_p^p \gtrsim N(E\cap I, 2^{-j})|I|^{-ps_p}. \tag{3.1}$$

By the definition of $\nu_E^\sharp$ in (1.4), this implies that (1.5) is sharp up to the ε-loss.

To this end, let I' be a subinterval of I with length $|I|/2$ such that $N(E\cap I', \delta) \geq \frac{1}{2}N(E\cap I,\delta)$ and let t_I be the boundary point of I such that $\operatorname{dist}(t_I, I') \geq |I|/4$; without loss of generality, we assume that t_I is the left endpoint.

Consider the radial function g_I given by

$$\widehat{g_I}(\xi) = e^{-it_I|\xi|}\varphi(2^{-j}|\xi|)$$

where φ is a nonnegative bump function on $(1/2, 2)$. We first observe

$$\|g_I\|_p \lesssim 2^{j(\frac{d+1}{2}-\frac{1}{p})}. \tag{3.2}$$

Indeed from Plancherel's theorem $\|g_I\|_2 \leq 2^{jd/2}$. For an L^∞ bound we observe first that a multiple integration-by-parts yields $|g_I(x)| = O(1)$ for $|x| < 1/2$ and $|x| \geq 3$. For $1/2 < |x| < 3$ we use the Fourier inversion formula for radial functions [SW71, §IV.3]

$$g_I(x) = (2\pi)^{-d/2}\int_0^\infty e^{-it_I s}\varphi(2^{-j}s)J_{\frac{d-2}{2}}(s|x|)(s|x|)^{-\frac{d-2}{2}}s^{d-1}\,\mathrm{d}s.$$

Recall also from [SW71, §IV.3] the well-known asymptotics for $|u| \geq 1$,

$$J_{\frac{d-2}{2}}(u) = \big(e^{-i(u-\frac{\pi}{4}(d-1))} + e^{i(u-\frac{\pi}{4}(d-1))}\big)(2\pi u)^{-1/2} + R(u) \tag{3.3}$$

where $|R(u)| = O(|u|^{-3/2})$ for $|u| \geq 1$. From this we see that $|g_I(x)| \lesssim 2^{j(d+1)/2}$, which is also the bound for $\|g_I\|_\infty$. Thus (3.2) follows using $\|g\|_p \leq \|g\|_2^{2/p}\|g\|_\infty^{1-2/p}$

for $2 \le p \le \infty$. We note that by a slightly more careful argument one can show the pointwise bound $|g_I(x)| \lesssim_N 2^{j\frac{d+1}{2}}(1+2^j||x|-t_I|)^{-N}$ which also gives (3.2).

We now turn to lower bounds for $\|e^{it\sqrt{-\Delta}}g_I\|_p$. Again by the Fourier inversion formula for radial functions we have

$$e^{it\sqrt{-\Delta}}g_I(x) = (2\pi)^{-d/2}\int_0^\infty e^{i(t-t_I)s}\varphi(2^{-j}s)J_{\frac{d-2}{2}}(s|x|)(s|x|)^{-\frac{d-2}{2}}s^{d-1}\,\mathrm{d}s.$$

Then, for $|x| \ge 2^{-j+2}$ we can write using (3.3)

$$e^{it\sqrt{-\Delta}}g_I(x) = T_t^- g_I(x) + T_t^+ g_I(x) + T_t^{\mathrm{rem}} g_I(x), \tag{3.4}$$

where

$$T_t^\pm g_I(x) = |x|^{-\frac{d-1}{2}}\frac{e^{\mp i\frac{\pi}{4}(d-1)}}{(2\pi)^{(d+1)/2}}\int_0^\infty e^{i(t-t_I\pm|x|)s}\varphi(2^{-j}s)s^{\frac{d-1}{2}}\,\mathrm{d}s$$

and the remainder term is given by

$$T_t^{\mathrm{rem}} g_I(x) = |x|^{-\frac{d-2}{2}}\int_0^\infty R(|x|s)\varphi(2^{-j}s)s^{\frac{d}{2}}\,\mathrm{d}s.$$

Given $t \in E_j \cap I'$, let $J_t = [t-t_I-2^{-j-5}, t-t_I+2^{-j-5}]$, and define $D_t = \{x : |x| \in J_t\}$. We will examine each of the terms in (3.4) for $x \in D_t$.

If $x \in D_t$, then

$$|T_t^- g_I(x)| \gtrsim |x|^{-\frac{d-1}{2}}\int_0^\infty \varphi(2^{-j}s)s^{\frac{d-1}{2}}\,\mathrm{d}s \gtrsim |x|^{-\frac{d-1}{2}}2^{j\frac{d+1}{2}}.$$

Consequently,

$$\|T_t^- g_I\|_{L^p(D_t)} \gtrsim 2^{j\frac{d+1}{2}}\Big(\int_{J_t} r^{-(d-1)(\frac{p}{2}-1)}\,\mathrm{d}r\Big)^{1/p} \gtrsim 2^{j(\frac{d+1}{2}-\frac{1}{p})}|I|^{-(d-1)(\frac{1}{2}-\frac{1}{p})} \tag{3.5}$$

using that if $r \in J_t$, then $r \sim |t-t_I| \sim |I|$.

For the term T_t^+, we have by repeated integration-by-parts,

$$|T_t^+ g_I(x)| \lesssim_N |x|^{-\frac{(d-1)}{2}}2^{j\frac{d-1}{2}}\frac{2^j}{\big(1+2^j(|x|+t-t_I)\big)^N}$$

for any $N > 0$. Thus

$$\begin{aligned}\|T_t^+ g_I\|_{L^p(D_t)} &\lesssim 2^{j\frac{d+1}{2}}\Big(\int_{J_t} r^{-(d-1)(\frac{p}{2}-1)}\frac{1}{(1+2^j r)^N}\,\mathrm{d}r\Big)^{1/p}\\ &\lesssim 2^{j\frac{d+1}{2}}|I|^{-(d-1)(\frac{1}{2}-\frac{1}{p})}(2^j|I|)^{-N/p}|J_t|^{\frac{1}{p}} \le M^{-1}2^{j(\frac{d+1}{2}-\frac{1}{p})}|I|^{-(d-1)(\frac{1}{2}-\frac{1}{p})}\end{aligned} \tag{3.6}$$

using that $r \sim |I|$ for $r \in J_t$, $|J_t| \sim 2^{-j}$, $(2^j|I|) \ge M$ and choosing $N > p$. For the remainder term we have for $x \in D_t$,

$$|T_t^{\mathrm{rem}} g_I(x)| \lesssim |x|^{-\frac{(d+1)}{2}}\int_0^\infty \varphi(2^{-j}s)s^{\frac{d-3}{2}}\,\mathrm{d}s \lesssim |x|^{-\frac{(d+1)}{2}}2^{j\frac{d+1}{2}}2^{-j}.$$

Then

$$\begin{aligned}\|T_t^{\mathrm{rem}} g_I\|_{L^p(D_t)} &\lesssim 2^{j\frac{d-1}{2}}\Big(\int_{J_t} r^{-(d-1)(\frac{p}{2}-1)} r^{-p}\,\mathrm{d}r\Big)^{\frac{1}{p}}\\ &\lesssim 2^{j\frac{d-1}{2}}|I|^{-(d-1)(\frac12-\frac1p)}|I|^{-1}|J_t|^{\frac1p} \lesssim M^{-1}2^{j(\frac{d+1}{2}-\frac1p)}|I|^{-(d-1)(\frac12-\frac1p)} \end{aligned} \tag{3.7}$$

since $r\sim|I|$ for $r\in I_t$ and $|I_t|\sim 2^{-j}$, $(2^j|I|)\geq M$. From (3.6) and (3.7) we obtain, with a sufficiently large choice of M,

$$\|T_t^+ g_I\|_{L^p(D_t)} + \|T_t^{\mathrm{rem}} g_I\|_{L^p(D_t)} \leq \frac12 \|T_t^- g_I\|_{L^p(D_t)}.$$

Combining this with (3.4) and the lower bound (3.5) for T_t^-, and taking the ℓ^p norm in $t\in E_j\cap I'$ we get

$$\Big(\sum_{t\in E_j\cap I'} \|e^{it\sqrt{-\Delta}} g_I\|_{L^p(D_t)}^p\Big)^{1/p} \gtrsim 2^{j(\frac{d+1}{2}-\frac1p)}|I|^{-(d-1)(\frac12-\frac1p)} N(E\cap I, 2^{-j})^{\frac1p};$$

here we used that $N(E\cap I', 2^{-j})\geq \frac12 N(E\cap I, 2^{-j})$. Since $\|g_I\|_p \lesssim 2^{j(\frac{d+1}{2}-\frac1p)}$ we obtain the desired lower bound (3.1).

4. LOWER BOUNDS FOR THE $L^p\to \ell^q_{E_j}(L^q)$ CONJECTURE

We construct counterexamples motivated by the examples for maximal operators in [AHRS21], [RS23]. These are associated to spherical pieces intermediate between spherical Knapp caps and full spheres. However, here we choose a sectorial localization on the Fourier side.

Proposition 4.1. *Let $1\leq p<\infty$, $q>p'$. Then there exist constants $c(q)$ and $\rho\ll 1$ such that for all intervals I with $2^j\geq 2^j|I|\geq\rho^{-1}$*

$$\sup_{\|f\|_p\leq 1}\Big(\sum_{t\in E_j}\|e^{it\sqrt{-\Delta}}f\|_q^q\Big)^{1/q} \geq c(q)\rho^{\frac{d}{q}}\frac{N(E\cap I,2^{-j})^{\frac1q}2^{j\frac{d+1}{2}(\frac1p-\frac1q)}}{|I|^{\frac{d-1}{2}(1-\frac1p-\frac1q)}}. \tag{4.1}$$

The case $p=2$ in the proposition shows that in Theorem 1.6 the critical $s_E(q)$ cannot be replaced by a smaller value for $r=q$, which implies the same conclusion for $2\leq r\leq q$ by the nesting of the ℓ^r spaces. The case for general p,q shows that in Conjecture 1.7 the exponent $s_E(p,q)$ cannot be replaced by a smaller one.

Proof (of Proposition 4.1). Let $m\in\mathbb{N}$ be such that $2^m\leq\rho^{-1}$ and $j\geq m$ and let I be an interval of length $2^{m-j}\leq|I|\leq 2^{m-j+1}$. Let I' be a subinterval of I such that $|I'|\approx\rho|I|$ and $N(E\cap I',2^{-j})\geq\rho N(E\cap I,2^{-j})$. Let $t_I\in I'\cap E_j$.

Let υ be a nonnegative $C_c^\infty(B(0,1))$ function such that $\upsilon=1$ in a neighborhood of the origin. Define $f_I\in L^p$ by its Fourier transform via

$$\widehat{f_I}(\xi) = (2\pi)^d\varphi(2^{-j}|\xi|)\upsilon(2^{\frac{m}{2}}(\tfrac{\xi}{|\xi|}-e_1))e^{-it_I|\xi|}.$$

We first show that

$$\|f_I\|_p \lesssim 2^{j(\frac{d+1}{2}-\frac{1}{p})}2^{-m\frac{d-1}{2p}}. \tag{4.2}$$

We use a standard argument from [SSS91] and decompose the Fourier transform into pieces supported on sectors of angular width $O(2^{-j/2})$. For this decomposition split variables as $\xi = (\xi_1, \xi')$ and note that on the support of $\widehat{f_I}$ we have $\xi_1 \approx 2^j$ and $|\xi'| \lesssim 2^{-m/2}\xi_1$. Choose $\chi \in C_c^\infty(\mathbb{R}^{d-1})$ supported in $(-1,1)^{d-1}$ such that $\sum_{\mathfrak{z}\in\mathbb{Z}^{d-1}} \chi(\xi' - \mathfrak{z}) = 1$ for all $\xi' \in \mathbb{R}^{d-1}$. Then we write $f_I = \sum_{\mathfrak{z}} f_{I,\mathfrak{z}}$ where

$$\widehat{f_{I,\mathfrak{z}}}(\xi) = (2\pi)^d e^{-it_I|\xi|}\varphi(2^{-j}|\xi|)\upsilon(2^{\frac{m}{2}}(\tfrac{\xi}{|\xi|} - e_1))\chi(2^{j/2}\tfrac{\xi'}{\xi_1} - \mathfrak{z}).$$

An integration-by-parts argument in [SSS91] gives

$$|f_{I,\mathfrak{z}}(x)| \lesssim_N \frac{2^j}{(1+2^j|\langle x, e_{\mathfrak{z}}\rangle - t_I|)^N}\frac{2^{j(d-1)/2}}{(1+2^{j/2}|\pi_{\mathfrak{z}}^\perp x|)^N} \tag{4.3}$$

where $e_{\mathfrak{z}} = \frac{(1,2^{-j/2}\mathfrak{z})}{\sqrt{1+2^{-j}|\mathfrak{z}|^2}}$ and $\pi_{\mathfrak{z}}^\perp$ is the orthogonal projection to the orthogonal complement of $\mathbb{R}e_{\mathfrak{z}}$. Here we use that $|\langle e_{\mathfrak{z}}, \nabla\rangle^N \upsilon(2^{\frac{m}{2}}(\frac{\xi}{|\xi|} - e_1))| \lesssim_N 2^{-jN}$ for $|2^{-j/2}\mathfrak{z}| \lesssim 2^{-m/2}$ and also $|\langle e_{\mathfrak{z}}, \nabla\rangle^N \chi(2^{j/2}\frac{\xi'}{\xi_1} - \mathfrak{z})| \lesssim_N 2^{-jN}$ for all $N \geq 0$.

One computes that $\|f_{I,\mathfrak{z}}\|_1 = O(1)$. In view of the support properties of $\widehat{f_I}$ the sum $\sum_{\mathfrak{z}} f_{I,\mathfrak{z}}$ extends over $O(2^{\frac{j-m}{2}(d-1)})$ contributing terms and thus we get $\|f_I\|_1 \lesssim 2^{\frac{j-m}{2}(d-1)}$ which is (4.2) for $p = 1$. Regarding $p = \infty$, we clearly have $\|f_{I,\mathfrak{z}}\|_\infty = O(2^{j\frac{d+1}{2}})$. However note that $t_I \approx 1$ and the vectors $t_I e_{\mathfrak{z}}$ are $c2^{-j/2}$-separated, and thus one can use the decay properties in (4.3) to see that the same bound holds for the sum, $\sum_{\mathfrak{z}} f_{I,\mathfrak{z}}$. That is, we get $\|f_I\|_\infty \lesssim 2^{j\frac{d+1}{2}}$ which is (4.2) for $p = \infty$. We now conclude (4.2) using $\|f\|_p \leq \|f\|_1^{1/p}\|f\|_\infty^{1-1/p}$ for $1 \leq p \leq \infty$.

For $t \in I' \cap E_j$, let

$$R_{I,t} = \big\{x = (x_1, x') \in \mathbb{R}^d : |x_1 + t - t_I| \leq 2^{-j},\ |x'| \leq \rho 2^{-j+\frac{m}{2}}\big\}.$$

We will next prove a lower bound for $|e^{it\sqrt{-\Delta}}f_I(x)|$ and $x \in R_{I,t}$.

We use polar coordinates in the Fourier variable ξ and write $\xi = r\theta(\omega)$ where $\omega \to \theta(\omega)$ is a smooth parametrization of S^{d-1} near e_1 with $\theta(0) = e_1$. Here, the parameter ω lives in a neighborhood of the origin of $\mathbb{R}^{d-1}$. Then

$$e^{it\sqrt{-\Delta}}f_I(x) = \int \upsilon(2^{\frac{m}{2}}(\theta(\omega) - e_1)) \int r^{d-1}\varphi(2^{-j}r)e^{ir(t-t_I+\langle x,\theta(\omega)\rangle)}\,\mathrm{d}r\,\mathrm{d}\sigma(\omega).$$

We write $\langle x, \theta(\omega)\rangle = x_1 + \langle x, \theta(\omega) - e_1\rangle$ and

$$e^{it\sqrt{-\Delta}}f_I(x) = \mathrm{I}(x,t) + \mathrm{II}(x,t)$$

where

$$\mathrm{I}(x,t) = \int r^{d-1}\varphi(2^{-j}r)e^{ir(t-t_I+x_1)}\,\mathrm{d}r \int \upsilon(2^{\frac{m}{2}}(\theta(\omega) - e_1))\,\mathrm{d}\sigma(\omega)$$

and

$$\mathrm{II}(x,t) = \int r^{d-1}\varphi(2^{-j}r)e^{ir(t-t_I+x_1)} \int u_m(r,\omega,x)\,\mathrm{d}r\,\mathrm{d}\sigma(\omega),$$

with

$$u_m(r,\omega,x) = \upsilon(2^{\frac{m}{2}}(\theta(\omega)-e_1))\big(e^{ir\langle x,\theta(\omega)-e_1\rangle}-1\big).$$

For the term $\mathrm{I}(x,t)$ we set $\phi(r)=\varphi(r)r^{d-1}$ and note

$$\mathrm{I}(x,t) = c_m 2^{-m\frac{d-1}{2}} 2^{jd}\widehat{\phi}(2^j(t_I-t-x_1))$$

with $c_m \approx 1$. Using that $(\int_{-1/4}^{1/4} |\widehat{\phi}(s)|^q\,\mathrm{d}s)^{1/q} \gtrsim 1$ we obtain the lower bound

$$\Big(\int_{R_{I,t}} |\mathrm{I}(x,t)|^q\,\mathrm{d}x\Big)^{1/q} \gtrsim \rho^{\frac{d-1}{q}} 2^{jd(1-\frac{1}{q})} 2^{-m\frac{d-1}{2}(1-\frac{1}{q})}. \tag{4.4}$$

For the term $\mathrm{II}(x,t)$ we get a corresponding upper bound, multiplied with an additional small factor of ρ. To see this, we expand

$$e^{ir\langle x,\theta(\omega)-e_1\rangle}-1 = \sum_{n=1}^{\infty}\frac{1}{n!}(ir)^n(\langle x,\theta(\omega)-e_1\rangle)^n \tag{4.5}$$

and write

$$\langle x,\theta(\omega)-e_1\rangle = x_1(\theta_1(\omega)-1) + \sum_{i=2}^{d} x_i\theta_i(\omega).$$

For $t\in I'$ and $x\in R_{I,t}$, we have $|x_1| \lesssim |t-t_I| \lesssim 2^{m-j}\rho$ and because of $\langle e_1, \frac{\partial\theta(\omega)}{\partial\omega_i}\rangle|_{\omega=0} = 0$ we get $|x_1(\theta_1(\omega)-1)| \lesssim 2^{-j}\rho$. Furthermore, for $i=2,\dots,d$, one has $|x_i\theta_i(\omega)| \lesssim 2^{-m/2}|x_i| \lesssim 2^{-j}\rho$. Thus,

$$|\langle x,\theta(\omega)-e_1\rangle| \lesssim 2^{-j}\rho \qquad \text{for } x\in R_{I,t},\ t\in I'. \tag{4.6}$$

Using the expansion (4.5) we write $\mathrm{II} = \sum_{n=1}^{\infty}\mathrm{II}_n$ and note the pointwise bounds

$$n!|\mathrm{II}_n(x,t)| \le 2^{j(d+n)}|\widehat{\phi}_n(2^j(t_I-t-x_1))| \int |\upsilon(2^{\frac{m}{2}}(\theta(\omega)-e_1)||\langle x,\theta(\omega)-e_1\rangle|^n\,\mathrm{d}\sigma(\omega),$$

where $\phi_n(r)=\varphi(r)r^{d-1+n}$. We have $|\widehat{\phi}_n(y)| \le C_d n^{d+1}(1+|y|)^{-d-1}$ as can be seen using a $(d+1)$-fold integration-by-parts in r. Hence

$$|\mathrm{II}_n(x,t)| \lesssim (C_d\rho)^n \frac{n^{d+1}}{n!} 2^{-m\frac{d-1}{2}} 2^{jd}(1+2^j|x_1+t-t_I|)^{-d-1} \qquad \text{for } x\in R_{I,t},$$

using also (4.6). Taking the $L^q(R_{I,t})$ norm and summing in $n\ge 1$ leads to

$$\Big(\int_{R_{I,t}} |\mathrm{II}(x,t)|^q\,\mathrm{d}x\Big)^{1/q} \lesssim_d \rho^{1+\frac{d-1}{q}} 2^{jd(1-\frac{1}{q})} 2^{-m\frac{(d-1)}{2}(1-\frac{1}{q})}. \tag{4.7}$$

Thus, for sufficiently small $\rho > 0$ we get from (4.2), (4.4), and (4.7)

$$\frac{\left(\sum_{t\in E_j\cap I'}\|e^{it\sqrt{-\Delta}}f_I\|_q^q\right)^{1/q}}{\|f_I\|_p} \gtrsim \rho^{\frac{d-1}{q}}\frac{N(E\cap I',2^{-j})^{\frac1q}2^{jd(1-\frac1q)}2^{-m\frac{d-1}{2}(1-\frac1q)}}{2^{j(\frac{d+1}{2}-\frac1p)}2^{-m\frac{d-1}{2p}}}.$$

The right-hand side equals $\rho^{\frac{d-1}{q}}N(E\cap I',2^{-j})^{\frac1q}2^{(m-j)\frac{d-1}{2}(\frac1p+\frac1q-1)}2^{j\frac{d+1}{2}(\frac1p-\frac1q)}$. Since $N(E\cap I',2^{-j})\geq \rho N(E\cap I,2^{-j})$, the lower bound (4.1) follows. □

5. Upper bounds in Theorem 1.1

Proposition 5.1. *Let $2\leq p<\frac{2d}{d-1}$ and $s_p=(d-1)(\frac12-\frac1p)$. Then for $\varepsilon>0$ and sufficiently large $j\geq 1$,*

$$\Big(\sum_{t\in E_j}\|e^{it\sqrt{-\Delta}}P_jf\|_p^p\Big)^{1/p}\lesssim_\varepsilon 2^{j\varepsilon}2^{j\frac1p\nu_E^\sharp(ps_p)}\|f\|_{L^p_{\mathrm{rad}}}. \tag{5.1}$$

Once this is proven we can use that $\frac1p\nu_E^\sharp(ps_p)=s_p$ for $p>p_\gamma$ and obtain (1.5) by interpolation (restricted to radial functions) of (5.1) with the $p=\infty$ version of the fixed-time estimate (1.1) for functions whose Fourier transform is supported in the annulus $\{|\xi|\approx 2^{-j}\}$.

Proof (of Proposition 5.1). For $0\leq m\leq j$ define

$$\kappa_{j,m}=\sup_{|I|=2^{m-j}}N(E\cap I)|I|^{-ps_p}$$

and note that, by definition of $\nu_E^\sharp$, we have $\kappa_{j,m}\leq C_\varepsilon 2^{j\varepsilon}2^{j\nu_E^\sharp(ps_p)}$ for any $\varepsilon>0$. It therefore suffices to prove that

$$\Big(\sum_{t\in E_j}\|e^{it\sqrt{-\Delta}}P_jf\|_p^p\Big)^{1/p}\lesssim\Big(\sum_{m=0}^j\kappa_{j,m}\Big)^{1/p}\|f\|_{L^p_{\mathrm{rad}}},\quad 2\leq p<\tfrac{2d}{d-1}. \tag{5.2}$$

It was proven in [MS95, Proposition 3.2] that for f radial, $|x|\geq 20$ and $t\in[1,2]$, the estimate

$$\Big(\int_{|x|\geq 20}|e^{it\sqrt{-\Delta}}P_jf(x)|^p\,\mathrm{d}x\Big)^{1/p}\lesssim\|f\|_{L^p_{\mathrm{rad}}}$$

holds for all $2\leq p<\infty$, with constant independent of t. Consequently,

$$\Big(\sum_{t\in E_j}\big\|e^{it\sqrt{-\Delta}}P_jf\big\|_{L^p(\mathbb{R}^d\setminus B(0,20))}^p\Big)^{1/p}\lesssim N(E,2^{-j})^{1/p}\|f\|_{L^p_{\mathrm{rad}}}.$$

Since $N(E,2^{-j})\leq\sup_{2^{-j}\leq|I|\leq1}N(E\cap I,2^{-j})|I|^{-ps_p}\lesssim\max_{0\leq m\leq j}\kappa_{j,m}$, the inequality (5.2) will follow from

$$\sum_{t\in E_j}\int_{|x|\leq 20}|e^{it\sqrt{-\Delta}}P_jf(x)|^p\,\mathrm{d}x\lesssim\sum_{m=0}^j\kappa_{j,m}\|f\|_{L^p_{\mathrm{rad}}}^p,\quad 2\leq p<\tfrac{2d}{d-1}. \tag{5.3}$$

Since $e^{it|\xi|}$ is a radial Fourier multiplier for fixed $t \in [1,2]$, and φ_j and f are radial, we can write the operator $e^{it\sqrt{-\Delta}}P_j$ as

$$e^{it\sqrt{-\Delta}}P_j f(x) = \int_0^\infty K_j(|x|,t,s) f_0(s)\,\mathrm{d}s$$

where $f(x) = f_0(|x|)$ and

$$K_j(r,s,t) = s^{d/2} r^{-(d-2)/2} \int_0^\infty J_{\frac{d-2}{2}}(\rho r) J_{\frac{d-2}{2}}(\rho s) \varphi(2^{-j}\rho) e^{it\rho}\,\mathrm{d}\rho\,.$$

See, for instance, [SW71, §IV.3]. Here $J_{\frac{d-2}{2}}$ denotes the Bessel function of order $\frac{d-2}{2}$. Using asymptotics of Bessel functions and integration-by-parts, it was shown in [MS95, Lemma 2.1] that these kernels satisfy the estimates

$$|K_j(r,s,t)| \lesssim \left(\frac{s}{r}\right)^{\frac{d-1}{2}} \sum_{\pm} \omega_j(t \pm r \pm s)$$

where $\omega_j(u) \lesssim_N 2^j(1+2^j|u|)^{-N}$ for all $N > 0$ and the sum is over all four choices of the two signs. Changing to polar coordinates and inserting the power weights into the function and operator, the inequality (5.3) follows from the unweighted one-dimensional estimates

$$\sum_{t\in E_j} \int_0^{20} r^{(d-1)(1-\frac{p}{2})} \Big| \int_0^\infty s^{(d-1)(\frac12-\frac1p)} \omega_j(t\pm r\pm s) f_0(s)\,\mathrm{d}s\Big|^p\,\mathrm{d}r$$

$$\lesssim \sum_{m=0}^{j} \kappa_{j,m} \int_0^\infty |f_0(s)|^p\,\mathrm{d}s \tag{5.4}$$

for all possible choices of $\pm$, where $2 \le p < \frac{2d}{d-1}$. For $p \ge 2$, we dominate the left-hand side in (5.4) by $\sum_{m=0}^{j} \mathrm{I}_m + \sum_{n\ge 10} \mathrm{II}_n$, where for $m < j$

$$\mathrm{I}_0 = \sum_{t\in E_j} \int_0^{2^{-j}} r^{(d-1)(1-\frac{p}{2})} \Big| \int_0^{2^{10}} s^{(d-1)(\frac12-\frac1p)} \omega_j(t\pm r\pm s) f_0(s)\,\mathrm{d}s\Big|^p\,\mathrm{d}r,$$

$$\mathrm{I}_m = \sum_{t\in E_j} \int_{2^{-j+m}}^{2^{-j+m+1}} r^{(d-1)(1-\frac{p}{2})} \Big| \int_0^{2^{10}} s^{(d-1)(\frac12-\frac1p)} \omega_j(t\pm r\pm s) f_0(s)\,\mathrm{d}s\Big|^p\,\mathrm{d}r,$$

$$\mathrm{I}_j = \sum_{t\in E_j} \int_1^{20} r^{(d-1)(1-\frac{p}{2})} \Big| \int_0^{2^{10}} s^{(d-1)(\frac12-\frac1p)} \omega_j(t\pm r\pm s) f_0(s)\,\mathrm{d}s\Big|^p\,\mathrm{d}r$$

and, with $n \ge 10$,

$$\mathrm{II}_n = \sum_{t\in E_j} \int_0^{20} r^{(d-1)(1-\frac{p}{2})} \Big| \int_{2^n}^{2^{n+1}} s^{(d-1)(\frac12-\frac1p)} \omega_j(t\pm r\pm s) f_0(s)\,\mathrm{d}s\Big|^p\,\mathrm{d}r.$$

The terms II_n can be treated in a straightforward manner. Note that, for $n \ge 10$, we have by Hölder's inequality

$$\mathrm{II}_n \lesssim \sum_{t\in E_j} \int_0^{20} r^{(d-1)(1-\frac{p}{2})} \Big[\int_{2^n}^{2^{n+1}} 2^{n(d-1)(\frac12-\frac1p)} 2^{-nN} 2^{-jN} |f_0(s)|\,\mathrm{d}s\Big]^p\,\mathrm{d}r$$

$$\lesssim \sum_{t\in E_j} \Big(\int_0^{20} r^{(d-1)(1-\frac{p}{2})}\,\mathrm{d}r\Big) 2^{-jNp} 2^{-nN'p} \int_1^\infty |f_0(s)|^p\,\mathrm{d}s$$
$$\lesssim N(E,2^{-j}) 2^{-jNp} 2^{-nN'p} \int_1^\infty |f_0(s)|^p\,\mathrm{d}s$$

for all $1 \le p < \frac{2d}{d-1}$ and $N' > 0$, provided N is chosen sufficiently large. Then

$$\sum_{n\ge 10} \mathrm{II}_n \lesssim 2^{-jN}\|f_0\|_p^p \tag{5.5}$$

for all $1 \le p < \frac{2d}{d-1}$ and any $N > 0$.

We now turn to the terms I_m. The term I_j is also trivial, since for $p \ge 2$

$$\mathrm{I}_j \lesssim \sum_{t\in E_j} \int_1^{20} \Big[\int_0^{2^{10}} \omega_j(t\pm r\pm s)|f_0(s)|\,\mathrm{d}s\Big]^p \mathrm{d}r \lesssim N(E,2^{-j})\|f_0\|_p^p \tag{5.6}$$

by Young's convolution inequality, noting that $\|\omega_j\|_1 \lesssim 1$. For the term I_0, we define for each $t \in [1,2]$ the interval $J_{t,j} = [t - 2^{-j+3}, t + 2^{-j+3}]$. We note that for $t \in [1,2]$ and $0 \le r \le 2^{-j}$

$$\int_0^{2^{10}} \omega_j(t\pm r\pm s)|f_0(s)|\mathbb{1}_{J_{t,j}^{\complement}}(s)\,\mathrm{d}s \lesssim 2^{-jN}\int_0^{2^{10}} |f_0(s)|\,\mathrm{d}s \lesssim 2^{-jN}\|f_0\|_p$$

for any $N > 0$, and

$$\int_0^{2^{10}} \omega_j(t\pm r\pm s)|f_0(s)|\mathbb{1}_{J_{t,j}}(s)\,\mathrm{d}s \lesssim \|\omega_j\|_{p'}\|f_0\mathbb{1}_{J_{t,j}}\|_p \lesssim 2^{j/p}\|f_0\mathbb{1}_{J_{t,j}}\|_p,$$

where both inequalities follow from the bound $\omega_j(u) \lesssim_N 2^j(1+2^j|u|)^{-N}$ for any $N > 0$ and Hölder's inequality. Using these,

$$\mathrm{I}_0 \lesssim \Big(\int_0^{2^{-j}} r^{(d-1)(1-\frac{p}{2})}\,\mathrm{d}r\Big)\Big(2^{-jNp}N(E,2^{-j})\|f_0\|_p^p + 2^j\sum_{t\in E_j}\|f_0\mathbb{1}_{I_{t,j}}\|_p^p\Big)$$
$$\lesssim 2^{j(d-1)(\frac{p}{2}-1)}\|f_0\|_p^p \lesssim 2^{j\nu_E^\sharp(ps_p)}\|f_0\|_p^p \tag{5.7}$$

for $2 \le p < \frac{2d}{d-1}$, using that $(d-1)(\frac{p}{2}-1) = ps_p \le \nu_E^\sharp(ps_p)$.

We now address the main terms with $0 < m < j$. We decompose $[1,2]$ into disjoint intervals $\{I_\mu\}$ of length $|I_\mu| = 2^{-j+m}$, and denote by I_μ^* the concentric interval with 5 times the length. Then

$$\mathrm{I}_m \lesssim 2^{(m-j)(d-1)(1-\frac{p}{2})}\sum_\mu \sum_{t\in E_j\cap I_\mu} \int_{2^{-j+m}}^{2^{-j+m+1}} \Big[\int_0^{2^{10}} \omega_j(t\pm r\pm s)|f_0(s)|\,\mathrm{d}s\Big]^p \mathrm{d}r$$
$$\lesssim \sup_{|I|=2^{m-j}} |I|^{-(d-1)(\frac{p}{2}-1)}\#(E_j\cap I)\sum_\mu \int_{I_\mu^*}\Big[\int_0^{2^{10}} \omega_j(r'\pm s)|f_0(s)|\,\mathrm{d}s\Big]^p \mathrm{d}r'$$
$$\lesssim \kappa_{j,m}\int_1^2 \Big[\int_0^{2^{10}} \omega_j(r'\pm s)|f_0(s)|\,\mathrm{d}s\Big]^p \mathrm{d}r' \lesssim \kappa_{j,m}\int_0^{2^{10}} |f_0(s)|^p\,\mathrm{d}s,$$

by the change of variables $r' = t \pm r$ and noting that $\|\omega_j\|_1 \lesssim 1$. Combining this estimate with (5.6) and (5.7) we get

$$\sum_{m=0}^{j} \mathrm{I}_m \lesssim \sum_{m=0}^{j} \kappa_{j,m} \|f\|_p^p . \tag{5.8}$$

By (5.8) and (5.5) we obtain (5.4) for all $2 \le p < \frac{2d}{d-1}$, which concludes the proof.□

6. UPPER BOUNDS IN THEOREM 1.6

For the upper bounds in Theorem 1.6 it suffices to settle the case $r = 2$. Setting $T_t^j = e^{it\sqrt{-\Delta}} P_j$ we get from Young's inequality and Plancherel's theorem

$$\|T_t^j\|_{L^2 \to L^q} \le 2^{jd(1/2-1/q)}, \quad 2 \le q \le \infty. \tag{6.1}$$

Moreover, by the usual TT^* argument [Str77], the asserted $L^2 \to L^q(\ell^2_{E_j})$ bound for $\{T_t^j\}_{t\in E_j}$ is equivalent with the inequality

$$\|S^j g\|_{L^q(\ell^2_{E_j})} \lesssim 2^{2js} \|g\|_{L^{q'}(\ell^2_{E_j})} \tag{6.2}$$

where

$$S^j g(y,t) = \sum_{t'\in E_j} T_t^j (T_{t'}^j)^* [g(\cdot,t')](y)$$

and $s > s_E(q)$. The Schwartz kernel of $T_t^j (T_{t'}^j)^*$ is given by $2^{jd}\mathcal{K}_j(y,t,y',t')$ where

$$\mathcal{K}_j(y,t,y',t') = \frac{1}{(2\pi)^d} \int_{\mathbb{R}^d} |\varphi(|\xi|)|^2 e^{i2^j[(t-t')|\xi| + \langle y-y',\xi\rangle]} \, \mathrm{d}\xi \, .$$

Let $\tilde\eta \in C_c^\infty(\mathbb{R}^d)$ with $\tilde\eta(w) = 1$ for $|w| \le 1/2$ and $\tilde\eta$ compactly supported in $\{|w| \le 1\}$. Let $\eta(w) = \tilde\eta(w) - \tilde\eta(2w)$. Now we set $\tilde\eta_{-j}(w) = \tilde\eta(2^j w)$ and $\eta_k(w) = \eta(2^{-k}w)$, so that $1 = \tilde\eta_{-j} + \sum_{m\ge 1} \eta_{m-j}$ for every j. We decompose

$$S^j = S_0^j + \sum_{m>0} S_m^j + \sum_{m>0} R_m^j \tag{6.3}$$

with

$$S_0^j g(y,t) = 2^{jd} \sum_{t'\in E_j} \int_{\mathbb{R}^d} \mathcal{K}_j(y,t,y',t') \tilde\eta_{-j}(y-y') g(y',t') \, \mathrm{d}y'$$

and, for $m \ge 1$

$$S_m^j g(y,t) = 2^{jd} \sum_{\substack{t'\in E_j \\ |t-t'|\le 2^{m-j+10}}} \int_{\mathbb{R}^d} \mathcal{K}_j(y,t,y',t') \eta_{m-j}(y-y') g(y',t') \, \mathrm{d}y',$$

$$R^j_m g(y,t) = 2^{jd} \sum_{\substack{t' \in E_j \\ |t-t'|>2^{m-j+10}}} \int_{\mathbb{R}^d} \mathcal{K}_j(y,t,y',t') \eta_{m-j}(y-y') g(y',t') \, \mathrm{d}y'.$$

The term S^j_0 is trivial, and the terms R^j_m and S^j_m with $m > j+10$ can be seen as error terms.

Lemma 6.1. *Let $2 \le q \le \infty$. For all $j \ge 0$, $N \ge 0$, we have the following bounds.*

(i) $\|S^j_0\|_{L^{q'}(\ell^2_{E_j}) \to L^q(\ell^2_{E_j})} \lesssim 2^{jd(1-2/q)}$.

(ii) For $m > j+10$, $\|S^j_m\|_{L^{q'}(\ell^2_{E_j}) \to L^q(\ell^2_{E_j})} \lesssim_N 2^{-(j+m)N}$.

(iii) For $m > 0$, $\|R^j_m\|_{L^{q'}(\ell^2_{E_j}) \to L^q(\ell^2_{E_j})} \lesssim_N 2^{-(j+m)N}$.

Part (i) follows from (6.1) and the Cauchy–Schwarz inequality. The proof of (ii) and (iii) is straightforward, and based on

$$|\mathcal{K}_j(y,t,y',t')| \lesssim_M \begin{cases} (1+2^j|t-t'|)^{-M} & \text{if } |t-t'| \ge 2|y-y'| \\ (1+2^j|y-y'|)^{-M} & \text{if } |y-y'| \ge 2|t-t'| \end{cases}$$

for any $M > 0$, which is obtained using integration-by-parts. We omit the details.

The main contribution comes from the terms S^j_m with $m \le j+10$.

Lemma 6.2. *Let $2 \le q \le \infty$ and*

$$\lambda_{j,m} = 2^{jd(1-\frac{2}{q})} 2^{-m(d-1)(\frac{1}{2}-\frac{1}{q})} \sup_{|I|=2^{m-j}} N(E \cap I, 2^{-j})^{\frac{2}{q}}.$$

For $m \le j+10$,

$$\|S^j_m g\|_{L^q(\ell^2_{E_j})} \lesssim \lambda_{j,m} \|g\|_{L^{q'}(\ell^2_{E_j})}. \tag{6.4}$$

Proof. By interpolation, it suffices to show (6.4) for $q = 2$ and $q = \infty$.

Case 1: $q = \infty$. After changing to polar coordinates, write $(2\pi)^d 2^{-jd} S^j_m g(y,t)$ as

$$\sum_{t' \in E_j} \int_{\mathbb{R}^d} \int_0^\infty \int_{S^{d-1}} |\varphi(r)|^2 r^{d-1} e^{i2^j r[(t-t')+\langle y-y',\theta\rangle]} g(y',t') \eta_{m-j}(y-y') \, \mathrm{d}\sigma(\theta) \, \mathrm{d}r \, \mathrm{d}y' \tag{6.5}$$

with $\mathrm{d}\sigma$ denoting the normalized surface measure on S^{d-1}. It is well-known [Ste93] that

$$\int_{S^{d-1}} e^{i\langle y,\theta\rangle} \, \mathrm{d}\sigma(\theta) = \sum_{\pm} e^{\pm i|y|} b_\pm(y)$$

for smooth symbols $b_\pm$ satisfying $\partial^\alpha b_\pm(w) \lesssim_\alpha |w|^{-\frac{d-1}{2}-|\alpha|}$ for $|w| \ge 1$ and $\alpha \in \mathbb{N}^d_0$. Then (6.5) becomes

$$\sum_{\pm} \sum_{t' \in E_j} \int_{\mathbb{R}^d} \int_0^\infty b_\pm(2^j r(y-y')) |\varphi(r)|^2 r^{d-1} e^{i2^j r[(t-t')\pm|y-y'|]} g(y',t') \eta_{m-j}(y-y') \, \mathrm{d}r \, \mathrm{d}y'.$$

Integrating by parts in r, we estimate

$$2^{-jd}|S^j_m g(y,t)| \lesssim 2^{-m\frac{d-1}{2}} \sum_{\pm} \sum_{t'\in E_j} \int_{\mathbb{R}^d} \big(1+2^j\big|t-t'\pm|y-y'|\big|\big)^{-N} |g(y',t')|\,\mathrm{d}y'$$

for any $N>0$. For fixed y, the $\ell^2(E_j)$ norm in t of this expression is bounded by

$$\begin{aligned} &2^{-m\frac{d-1}{2}} \sum_{\pm} \int_{\mathbb{R}^d} \Big(\sum_{t\in E_j} \Big| \sum_{t'\in E_j} \big(1+2^j\big|t-t'\pm|y-y'|\big|\big)^{-N} g(y',t')\Big|^2\Big)^{1/2} \mathrm{d}y' \\ &\lesssim 2^{-m\frac{d-1}{2}} \sum_{\pm} \int_{\mathbb{R}^d} \Big(\sum_{t'\in E_j} |g(y',t')|^2\Big)^{1/2} \mathrm{d}y' \end{aligned}$$

where we applied Schur's test on the 1-separated set $\{2^j t : t\in E_j\}$. Combining the above we get

$$|S^j_m g(y,\cdot)|_{\ell^2_{E_j}} \lesssim 2^{jd} 2^{-m\frac{d-1}{2}} \|g\|_{L^1(\ell^2_{E_j})}$$

which yields (6.4) with $q=\infty$.

Case 2: $q=2$. Using the Fourier inversion theorem for $\eta_{m-j}(y-y')$ we write

$$S^j_m g(y,t) = \frac{1}{(2\pi)^d} \int_{\mathbb{R}^d} \widehat{\eta}(\omega) e^{i2^{j-m}\langle y,\omega\rangle} \sum_{\substack{t'\in E_j\\ |t-t'|\le 2^{m-j+10}}} T^j_t (T^j_{t'})^* \big[g(\cdot,t') e^{-i2^{j-m}\langle y',\omega\rangle}\big]\,\mathrm{d}\omega.$$

In view of the rapid decay of $\widehat{\eta}(\omega)$, the inequality (6.4) for $q=2$ follows via Minkowski's inequality from

(6.6)
$$\Big\| \Big(\sum_{t'\in E_j} \Big| \sum_{\substack{t\in E_j\\ |t-t'|\le 2^{m-j+10}}} T^j_t (T^j_{t'})^*[g(\cdot,t')]\Big|^2\Big)^{1/2}\Big\|_2 \lesssim \sup_{|I|=2^{m-j}} N(E\cap I, 2^{-j}) \|g\|_{L^2(\ell^2_{E_j})}.$$

For $\mu\in\mathbb{Z}$ we let $I_\mu = [\mu 2^{m-j}, (\mu+1)2^{m-j}]$. The left-hand side above is equal to

$$\begin{aligned} &\Big(\sum_{\mu} \sum_{t\in E_j\cap I_\mu} \int_{\mathbb{R}^d} \Big| \sum_{\substack{t'\in E_j\\ |t-t'|\le 2^{m-j+10}}} T^j_t (T^j_{t'})^*[g(\cdot,t')](y)\Big|^2 \mathrm{d}y\Big)^{\frac12} \\ &\lesssim \Big(\sum_{\substack{(\mu,\mu')\\ |\mu-\mu'|\le 2^{11}}} N(E_j\cap I_{\mu'}, 2^{-j}) \sum_{t\in E_j\cap I_\mu} \sum_{t'\in E_j\cap I_{\mu'}} \int |T^j_t (T^j_{t'})^*[g(\cdot,t')](y)|^2\,\mathrm{d}y\Big)^{\frac12} \\ &\lesssim \sup_{|I|=2^{m-j}} N(E\cap I, 2^{-j})^{\frac12} \Big(\sum_{t'\in E_j} \sum_{\substack{t\in E_j\\ |t-t'|\lesssim 2^{m-j}}} \|g(\cdot,t')\|_2^2\Big)^{1/2} \\ &\lesssim \sup_{|I|=2^{m-j}} N(E\cap I, 2^{-j}) \|g\|_{L^2(\ell^2_{E_j})} \end{aligned}$$

where we have used Cauchy–Schwarz in the first inequality and $\|T^j_t\|_{2\to 2}=1$ in the second inequality. Thus (6.6) follows. This finishes the proof of Lemma 6.2. □

Proof (of Theorem 1.6, upper bounds). As discussed at the beginning of this section, the inequality (1.12) follows from (6.2). To prove the latter, we use the decomposition (6.3) and the triangle inequality. By Lemma 6.1

$$\|S_0^j g\|_{L^q(\ell^2_{E_j})} + \sum_{m\geq j+10} \|S_m^j g\|_{L^q(\ell^2_{E_j})} + \sum_{m\geq 0} \|R_m^j g\|_{L^q(\ell^2_{E_j})} \lesssim 2^{jd(1-\frac{2}{q})}\|g\|_{L^{q'}(\ell^2_{E_j})}$$

Since $\nu_E^\sharp(\alpha)\geq \alpha$, we have $2^{jd(1-\frac{2}{q})}\leq 2^{j(d+1)(\frac12-\frac1q)}2^{\frac{2j}{q}\nu_E^\sharp(\frac{d-1}{2}(\frac{q}{2}-1))} = 2^{2js_E(q)}$ and thus the above bound is admissible towards proving (6.2).

We next turn to the terms S_m^j with $m<j+10$. By the definitions of $\lambda_{j,m}$, $\nu_E^\sharp$ and $s_E(q)$ we get for $\varepsilon>0$

$$\begin{aligned}\lambda_{j,m} &\leq 2^{j(d+1)(\frac12-\frac1q)}\big[\sup_{|I|=2^{m-j}} |I|^{-(d-1)(\frac{q}{2}-1)}N(E\cap I,2^{-j})\big]^{\frac2q}\\ &\lesssim_\varepsilon 2^{j(d+1)(\frac12-\frac1q)}2^{\frac{2j}{q}(\nu_E^\sharp(\frac{d-1}{2}(\frac{q}{2}-1))+\varepsilon)} \leq 2^{j(2s_E(q)+\varepsilon)}.\end{aligned}$$

Thus, by Lemma 6.2 we obtain for $s>s_E(q)$

$$\sum_{m\leq j+10}\|S_m^j g\|_{L^q(\ell^2_{E_j})} \lesssim_\varepsilon (1+j)2^{j(2s_E(q)+\varepsilon)}\|g\|_{L^{q'}(\ell^2_{E_j})} \lesssim_s 2^{2js}\|g\|_{L^{q'}(\ell^2_{E_j})}$$

which concludes the proof of (6.2). □

The above argument can also recover a sharper $L^2\to L^q$ result for the square-function associated with a set of given Assouad dimension that was previously proved independently by Wheeler [Whe24+], and by the first, second and fourth author (unpublished). Recall that $\dim_{\mathrm{A}} E$ is the infimum over all exponents a such that $N(E\cap I,\delta)\leq C_a(|I|/\delta)^a$ holds for all $\delta\in(0,1)$ and all intervals I with $\delta\leq|I|\leq 1$.

Proposition 6.3. *Let $2\leq r\leq q<\infty$, $\gamma_\circ=\dim_{\mathrm{A}} E$ and $q_\circ=\frac{2(d-1+2\gamma_\circ)}{d-1}$.*

(i) For $q>q_\circ$,

$$\Big\|\Big(\sum_{t\in E_j}|e^{-it\sqrt{-\Delta}}P_jf|^r\Big)^{1/r}\Big\|_q \lesssim 2^{jd(\frac12-\frac1q)}\|f\|_2.$$

(ii) Suppose $\sup_{0<\delta<|I|\leq 1}\big(\frac{\delta}{|I|}\big)^{\gamma_\circ}N(E\cap I,\delta)<\infty$. Then

$$\Big\|\Big(\sum_{t\in E_j}|e^{-it\sqrt{-\Delta}}P_jf|^r\Big)^{1/r}\Big\|_{L^{q_\circ,\infty}} \lesssim 2^{jd(\frac12-\frac{1}{q_\circ})}\|f\|_2.$$

Proof. It suffices to show the $r=2$ case.

For part (i) we have $\lambda_{j,m}\lesssim_\varepsilon 2^{jd(1-\frac2q)}2^{-m((d-1)(\frac12-\frac1q)-\frac{2\gamma_\circ}{q}+\varepsilon)}$ and thus

$$\|S_m^j g\|_{L^q(\ell^2_{E_j})} \lesssim_\varepsilon 2^{jd(1-\frac2q)}2^{-m((d-1)(\frac12-\frac1q)+\frac{2\gamma_\circ}{q}+\varepsilon)}\|g\|_{L^{q'}(\ell^2_{E_j})}. \tag{6.7}$$

Observe that $(d-1)(\frac12-\frac1q)-\frac{2\gamma_\circ}{q}>0$ for $q>q_\circ$, hence we may sum in $m\leq j+10$ in this range and get part (i) by the usual TT^* argument used above.

For part (ii), we use the stronger assumption $N(E \cap I, \delta) \lesssim (\delta/|I|)^{-\gamma_\circ}$ to get (6.7) with $\varepsilon = 0$. Then Bourgain's interpolation trick [Bou85; CSWW99] yields

$$\Big\| \sum_{m \le j+10} S^j_m g \Big\|_{L^{q_\circ,\infty}(\ell^2_{E_j})} \lesssim 2^{jd(1-\frac{2}{q_\circ})} \|g\|_{L^{q_\circ',1}(\ell^2_{E_j})}.$$

Now part (ii) follows again by a TT^* argument, using duality for vector-valued Lorentz spaces. □

ACKNOWLEDGEMENTS

D.B., J.R. and A.S. were supported through the program *Oberwolfach Research Fellows* by Mathematisches Forschungsinstitut Oberwolfach in 2023. D.B. was supported in part by the AEI grants RYC2020-029151-I and PID2022-140977NA-I00. J.R. was supported in part by NSF grant DMS-2154835. J.R. also thanks the Hausdorff Research Institute for Mathematics in Bonn for providing a pleasant working environment during the Fall 2024 trimester program. A.R. was supported by Tuomas Orponen's grant from the Research Council of Finland via the project Approximate Incidence Geometry, grant no. 355453. A.S. was supported in part by NSF grant DMS-2348797.

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