Frostman's lemma and subadditive functions

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ABSTRACT. We give an exposition of Frostman's lemma from the perspective of subadditive functions on trees.

1. FROSTMAN'S LEMMA

Let $E \subset \mathbb{R}^d$ be an arbitrary set. The *Hausdorff s-content* of E is the quantity

$$\mathcal{H}^s_\infty(E) = \inf\Big\{\sum_i |E_i|^s : E \subset \bigcup_i E_i\Big\}.$$

Here, the infimum is over all families of sets $\{E_i\}$ and $|E_i|$ denotes the diameter of the set E. The Hausdorff content is countably subadditive: if $E \subset \bigcup E_i$, then

$$\mathcal{H}^s_\infty(E) \leq \sum_i \mathcal{H}^s_\infty(E_i).$$

On the other hand, Hausdorff content is not even finitely additive on disjoint sets.

The Hausdorff content is a lower bound for Hausdorff measure, and moreover $\mathcal{H}^s_\infty(E) = 0$ if and only if $\mathcal{H}^s(E) = 0$. In particular, the Hausdorff dimension can be defined purely in terms of Hausdorff content as $\dim_{\mathrm{H}} E = \inf\{s : \mathcal{H}^s_\infty(E) = 0\}$.

Obtaining upper bounds on Hausdorff content involves finding optimal covers, whereas finding lower bounds on Hausdorff content requires bounding the cost of all covers. A convenient way to obtain such bounds is to define measures on E which in some meaningful sense respect the geometry of E.

A particularly robust notion of s-dimensionality for measures is the following. We say that a Borel measure μ is *s-Frostman* if for all $x \in \mathbb{R}^d$ and $r > 0$,

$$\mu(B(x,r)) \leq r^s.$$

A classical observation, often called the *mass distribution principle*, is that the existence of Frostman measures provides a lower bound on the Hausdorff content.

Lemma 1.1. *Let $E \subset \mathbb{R}^d$ be Borel and suppose μ is s-Frostman. Then*

$$\mathcal{H}^s_\infty(E) \geq 2^{-d} \cdot \mu(E).$$

Proof. Let $\{E_i\}_i$ be any cover for E. Then since each set E_i is contained in a ball $B(x_i, |E_i|)$,

$$\mu(E) \leq \sum_i \mu(E_i) \leq 2^d |E_i|^s.$$

Since $\{E_i\}_i$ was arbitrary, by rearranging we obtain the desired bound. $\square$

Frostman's lemma is a fundamental theorem in geometry which states that the converse is also true. This result was first established in Otto Frostman's PhD thesis [Fro35].

Theorem 1.2 (Frostman's lemma). *Let $E \subset \mathbb{R}^d$ be compact with $\mathcal{H}^s_\infty(E) > 0$. Then there exists a s-Frostman measure μ with $\mu(E) \geq 2^{-d}\mathcal{H}^s_\infty(E)$.*

A generalization of E for analytic sets also holds; see for instance the exposition in [BP17, Appendix B].

The goal of this note is to give an exposition of the proof of Theorem 1.2 from the perspective of subadditive functions on trees. This proof is of a similar flavour to that given by Tolsa [Tol14, Theorem 1.23]. Beyond a proof of Theorem 1.2, we also hope to answer the following questions:

- Why does the Hausdorff s-content appear?
- Can we give a meaningful description of the set of all s-Frostman measures?

We will demonstrate the universality of Hausdorff content and give a simple inductive description of all s-Frostman measures on trees.

1.1. Trees and tree-valued functions. Instead of working with compact subsets $E \subset \mathbb{R}^d$, we it simpler to work instead with representations of the sets E by compact ultrametric spaces which we call *metric trees*. By taking a representation of $E \subset \mathbb{R}^d$ using a tree (such as the dyadic tree) it will not be so difficult to transfer our results from trees back to the original set.

Fix a number $M \in \mathbb{N}$ and $\xi \in (0, 1)$ and consider the space $\Omega = \{1, \ldots, M\}^{\mathbb{N}}$ equipped with the metric

$$d(x, y) = \inf\{\xi^m : x_1 \ldots x_m = y_1 \ldots y_m\}.$$

Given a finite word $\mathtt{i} \in \{1, \ldots, M\}^m$, we write $|\mathtt{i}| = m$ and

$$[\mathtt{i}] = \{x \in \Omega : x_1 \ldots x_m = \mathtt{i}\} \subset \Omega.$$

The metric d is precisely such that the sets $[\mathtt{i}]$ are open and closed balls with diameter ξ^m. In fact, each closed ball $B(x, r) = [\mathtt{i}]$ where $\mathtt{i}$ is the maximal finite prefix of x with $\xi^{|\mathtt{i}|} \geq r$

Now, let $K \subset \Omega$ be non-empty and compact. We associate with the compact set K a tree $\mathcal{T} \subset \{1, \ldots, M\}^*$ defined by the rule

$$\mathcal{T} = \bigcup_{n=0}^{\infty} \mathcal{T}_m \quad \text{where} \quad \mathcal{T}_m = \{\mathtt{i} \in \{1, \ldots, M\}^M : [\mathtt{i}] \cap K \neq \varnothing\}.$$

Since K is compact, it holds that

$$K = \bigcap_{m=0}^{\infty} \bigcup_{\mathtt{i} \in \mathcal{T}_m} [\mathtt{i}].$$

We now introduce our key definitions.

Definition 1.3. Let $\rho\colon \mathcal{T} \to [0,\infty)$ be some function. We say that ρ is *subadditive* if for all $\mathtt{i} \in \mathcal{T}$,

$$\rho(\mathtt{i}) \leq \sum_{\substack{j=1,\ldots,M \\ \mathtt{i}j\in\mathcal{T}}} \rho(\mathtt{i}j).$$

We say that ρ is *additive* if equality holds for all $\mathtt{i} \in \mathcal{T}$ in the above equation.

Of course, iterating the definition of subadditivity yields the following: if $\mathtt{j}$ is arbitrary and $[\mathtt{i}_k]_k$ is a finite cover of $[\mathtt{j}] \cap K$, then

$$\rho(\mathtt{j}) \leq \sum_k \rho(\mathtt{i}_k). \tag{1.1}$$

Also, there is a one-to-one correspondence between additive functions α on $\mathcal{T}$ and finite Borel measures on K: firstly, given a measure μ, the assignment

$$\alpha(\mathtt{i}) = \mu([\mathtt{i}]) \tag{1.2}$$

is additive; and conversely it is well-known that given an additive function α there necessarily exists a unique Borel measure μ satisfying (1.2).

Let us introduce one more definition.

Definition 1.4. We say that a subset $\Delta \subset \mathcal{T}$ is a *cut-set* if each $x \in K$ has exactly one prefix in Δ. We then let $\mathcal{T}_\Delta$ denote the set of all finite prefixes of words in Δ.

We will prove the following generalization of Frostman's lemma.

Theorem 1.5. *Let $f\colon \mathcal{T} \to [0,\infty)$ be any function. Then there exists a unique maximal subadditive function $\kappa \leq f$ on $\mathcal{T}$. Moreover, if $\Delta \subset \mathcal{T}$ is a cut-set and $\alpha_0\colon \mathcal{T}_\Delta \to [0,\infty)$ is an additive function with $\alpha_0 \leq \kappa$, then α_0 extends to an additive function $\alpha \leq \kappa$ on $\mathcal{T}$.*

Before we continue with the proof of this theorem, let us briefly explain how this relates to Frostman's lemma. Fix a compact set K with associated tree $\mathcal{T}$, and let $s \geq 0$. Let $f_s\colon \mathcal{T} \to [0,\infty)$ denote the function $\mathtt{i} \mapsto r^{|\mathtt{i}|s}$.

Definition 1.6. We say that a function $f\colon \mathcal{T} \to [0,\infty)$ is *s-Frostman* if $f \leq f_s$.

Given an exponent s, and recalling the correspondence between additive functions and measures, our goal is to find a non-zero additive s-Frostman function. It will turn out that the subadditive function κ corresponding to f_s is precisely the Hausdorff content $\kappa(\mathtt{i}) = \mathcal{H}^s_\infty([\mathtt{i}]\cap K)$, and the function α is exactly the s-Frostman measure which can be taken to be non-zero if and only if $\kappa(\varnothing) = \mathcal{H}^s_\infty(K) > 0$.

Remark 1.7. Of course, there is nothing particularly special about the potential $f_s(\mathtt{i}) = \xi^{|\mathtt{i}|s}$. It is quite common, for instance, to consider a general *gauge function* φ (that is, an increasing function $\varphi\colon [0,\infty) \to [0,\infty)$ with $\varphi(0) = 0$) and define $f_\varphi(\mathtt{i}) = \varphi(\xi^{|\mathtt{i}|})$. Since there are no required assumptions on the function f in Theorem 1.5, the theory works in an analogous way.

1.2. Hausdorff content and maximal subadditive functions. We first show, given a general function $f\colon \mathcal{T} \to [0,\infty)$, that there is a unique maximal subadditive function bounded above by f.

Lemma 1.8. *Let $f\colon \mathcal{T} \to [0,\infty)$ be any function and define $\kappa\colon \mathcal{T} \to [0,\infty)$ by the rule*

$$\kappa(\mathtt{j}) = \inf\Big\{\sum_{k=1}^{m} f(\mathtt{i}_k) : m \in \mathbb{N},\ [\mathtt{j}] \cap K \subset \bigcup_{k=1}^{m} [\mathtt{i}_k],\ [\mathtt{i}_k] \subset [\mathtt{j}]\Big\}. \tag{1.3}$$

Then κ is the unique maximal subadditive function with $\kappa \leq f$.

Proof. To verify that κ is indeed subadditive, let $[\mathtt{j}_\ell]_{\ell=1}^m$ be any finite collection of cylinders and let $\varepsilon > 0$ be arbitrary. For each ℓ, let $[\mathtt{i}_{k,l}]_k$ be a finite collection of cylinders covering $[\mathtt{j}_\ell]$ such that

$$\kappa(\mathtt{j}_\ell) \geq \sum_k f(\mathtt{i}_{k,\ell}) - \varepsilon.$$

Then since $\{[\mathtt{i}_{k,\ell}]\}_{k,\ell}$ is a cover for $[\mathtt{j}]$,

$$\sum_{\ell=1}^{m} \kappa(\mathtt{j}_\ell) \geq \sum_k \sum_\ell f(\mathtt{i}_{k,\ell}) - m\varepsilon \geq \kappa(\mathtt{j}) - m\varepsilon.$$

Since $\varepsilon > 0$ was arbitrary, it follows that κ is subadditive.

To observe that κ is maximal, let $\rho\colon \mathcal{T} \to [0,\infty)$ be any subadditive function with $\rho \leq f$. Let $\mathtt{j} \in \mathcal{T}$ be arbitrary and let $[\mathtt{i}_k]_k$ be a finite cover for $[\mathtt{j}] \cap K$. Then by subadditivity and the upper bound by f

$$\rho(\mathtt{j}) \leq \sum_k \rho(\mathtt{i}_k) \leq \sum_k f(\mathtt{i}_k).$$

But $\mathtt{i}_k$ was an arbitrary finite cover of $[\mathtt{j}] \cap K$, so recalling the definition of κ, $\rho(\mathtt{j}) \leq \kappa(\mathtt{j})$ as claimed. □

The definition of Hausdorff content from the introduction holds in arbitrary metric spaces since it only requires the notion of the diameter of a set. In our setting, since every ball $B(x,r)$ is of the form $[\mathtt{i}]$ for some finite word $\mathtt{i}$ and each $[\mathtt{i}]$ has diameter $\xi^{|\mathtt{i}|}$, it reduces to the following:

$$\mathcal{H}^s_\infty(E) = \inf\Big\{\sum_i \xi^{|\mathtt{i}|} : E \subset \bigcup_i [\mathtt{i}]\Big\}.$$

Since the function f_s is decreasing (that is, if $[\mathtt{i}] \subset [\mathtt{j}]$ then $f_s(\mathtt{i}) \leq f_s(\mathtt{j})$), applying Theorem 1.8, the unique maximal subadditive function $\kappa_s \leq f_s$ is exactly given by

$$\kappa_s(\mathtt{i}) = \mathcal{H}^s_\infty([\mathtt{i}] \cap K).$$

In this language, the mass distribution principle is the following fact: the weaker property of a subadditive function ρ being s-Frostman necessarily implies that ρ is bounded above by κ_s. Unlike in the Euclidean case, there is no loss of constant.

Corollary 1.9. *Let $K \subset \Omega$ be compact. Then κ_s is the unique maximal subadditive s-Frostman function on $\mathcal{T}$.*

1.3. Additive functions bounded above by subadditive functions. We now prove the second half of Theorem 1.5.

Proposition 1.10. *Let $K \subset \Omega$ be compact and let ρ be subadditive on the associated tree $\mathcal{T}$. If $\Delta \subset \mathcal{T}$ is a cut-set and $\alpha_0 \colon \mathcal{T}_\Delta \to [0, \infty)$ is an additive function with $\alpha_0 \leq \rho$, then α_0 extends to an additive function $\alpha \leq \rho$ on $\mathcal{T}$.*

Proof. We inductively define a function $\alpha \leq \rho$ satisfying the hypotheses. Begin by setting $\alpha = \alpha_0$ on $\mathcal{T}_\Delta$; in particular $\alpha(\varnothing)$ is already defined.

Now, suppose we have defined $\alpha(\mathtt{i})$ but not any children of $\mathtt{i}$. Let $J \subset \{1, \ldots, M\}$ denote the indices j such that $\mathtt{i}j \in \mathcal{T}$. We must choose $\alpha(\mathtt{i}j)$ for $j \in J$ such that the hypotheses hold:

(i) $\sum_{j \in J} \alpha(\mathtt{i}j) = \alpha(\mathtt{i})$; and
(ii) $\alpha(\mathtt{i}j) \leq \rho(\mathtt{i}j)$.

These conditions are compatible since by induction and subadditivity of ρ

$$\alpha(\mathtt{i}) \leq \rho(\mathtt{i}) \leq \sum_{j \in J} \rho(\mathtt{i}j).$$

Thus the construction may continue, completing the proof. $\square$

Remark 1.11. In the above proof, one might set

$$\alpha(\mathtt{i}j) = \alpha(\mathtt{i}) \cdot \frac{\rho(\mathtt{i}j)}{\sum_{k \in J} \rho(\mathtt{i}k)} \tag{1.4}$$

which clearly satisfies (i); and by induction, using $\alpha(\mathtt{i}) \leq \rho(\mathtt{i})$,

$$\alpha(\mathtt{i}j) \leq \rho(\mathtt{i}) \cdot \frac{\rho(\mathtt{i}j)}{\sum_{k \in J} \rho(\mathtt{i}k)} \leq \rho(\mathtt{i}j)$$

where the second inequality follows since ρ is subadditive. This is the choice made in Tolsa's proof of Frostman's lemma in [Tol14, Theorem 1.23].

The choice (1.4) is the only choice if and only if $\alpha(\mathtt{i}) = \sum_{k \in J} \rho(\mathtt{i}k)$.

This completes the proof of our main result.

Proof (of Theorem 1.5). Let $f \colon \mathcal{T} \to [0, \infty)$ be any function. Then Theorem 1.8 guarantees the existence of a unique maximal subadditive function $\rho \leq f$, and the theorem follows by applying Theorem 1.10 to κ. $\square$

In particular, we obtain Frostman's lemma as a direct consequence.

Corollary 1.12. *Let $K \subset \Omega$ be compact. Then*

$$\mathcal{H}^s_\infty(K) = \max\{\mu(K) : \mu \text{ is } s\text{-Frostman}\}.$$

Proof. By Theorem 1.9, if μ is s-Frostman, then $\mu(K) \leq \kappa_s(\varnothing) = \mathcal{H}^s_\infty(K)$. Conversely, let $\Delta = \{\varnothing\}$ and define $\alpha_0(\varnothing) = \rho(\varnothing)$, which is trivially additive. Applying Theorem 1.10, α_0 extends to an additive function α with $\alpha(\varnothing) = \alpha_0(\varnothing)$ and $\alpha \leq \rho \leq f_s$. Then the associated measure μ is s-Frostman and has $\mu(K) = \alpha(\varnothing) = \rho(\varnothing) = \mathcal{H}^s_\infty(K)$, as claimed. $\square$

In fact, the proof of Theorem 1.10 gives an *inductive* description of all s-Frostman measures μ. The property

$$\mu([\mathtt{i}]) \leq \mathcal{H}^s_\infty([\mathtt{i}] \cap K) \tag{1.5}$$

is the *only* obstruction to being s-Frostman: having defined $\mu([\mathtt{i}])$ for words $|\mathtt{i}| \leq m$ with the property that (1.5) holds, any definition of $\mu([\mathtt{i}j])$ for $[\mathtt{i}j] \cap K \neq \varnothing$ satisfying (1.5) is the restriction of some s-Frostman measure. Every s-Frostman measure on K can be obtained by following the algorithm in the proof of Theorem 1.10.

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